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MARCINKIEWICZ INTEGRAL WITH ROUGH KERNELS ON GENERALIZED WEIGHTED MORREY SPACES

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Abstract: In this paper, we study the boundedness of the Marcinkiewicz operators μ_Ω with rough kernels $\Omega \in L_s(S^{n-1})$ for some $s \in (1,\infty]$ on generalized weighted Morrey spaces $M_{p,\varphi}(\omega)$. We find the sufficient conditions on the pair (φ_1,φ_2) with $s' < p < \infty$ and $\omega \in A_{p/s'}$ or $1 < p < s$ and $\omega^{1-p'} \in A_{p'/s'}$ which ensures the boundedness of the operators μ_Ω from one generalized weighted Morrey space $M_{p,\varphi_1}(\omega)$ to another $M_{p,\varphi_2}(\omega)$.

Keywords: Marcinkiewicz operator; rough kernel; generalized weighted Morrey spaces; $A_p(R^n)$ weights.

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1. Introduction.

The classical Morrey spaces were originally introduced by Morrey in [18] to study the local behavior of solutions to second order elliptic partial differential equations. Guliyev, Mizuhara and Nakai [8, 17, 20] introduced generalized Morrey spaces $M^{p,\phi}(\mathbb{R}^n)$ (see, also [9, 21]). Recently, Komori and Shirai [16] considered the weighted Morrey spaces $L^{p,k}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderon-Zygmund operator on these spaces. Guliyev [10] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(\omega)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,k}(\omega)$. In [10] Guliyev also studied the boundedness of the classical operators and their commutators in these spaces $M_{p,\varphi}(\omega)$, see also [3, 11, 12, 13].

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of R^n $(n \ge 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on R^n. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t>0$ and $x \in R^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| F_{\Omega,t}(f)(x) \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function.

We find the sufficient conditions on the pair (φ_1, φ_2) with $s' < p < \infty$ and $\omega \in A_{p/s'}$ or $1 < p < s$ and $\omega^{1-p'} \in A_{p'/s'}$ which ensures the boundedness of the operators μ_Ω from one generalized weighted Morrey space $M_{p,\varphi_1}(\omega)$ to another $M_{p,\varphi_2}(\omega)$.

By A $\lesssim$ B we mean that A $\le$ CB with some positive constant C independent of appropriate quantities. If A $\lesssim$ B and B $\lesssim$ A, we write A $\approx$ B and say that A and B are equivalent.

2. Preliminaries.

We recall that a weight function w is in the Muckenhoupt's class $A_p(\mathbb{R}^n)$ [19], $1 < p < \infty$, if

$$[w]_{A_p} := \sup_B [w]_{A_p(B)}$$
$$= \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1}, \tag{2.1}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B by Hölder's inequality

$$[w]_{A_p(B)}^{1/p} = |B|^{-1} \|w\|_{L_1(B)}^{1/p} \left\| w^{-1/p} \right\|_{L_{p'}(B)} \ge 1. \tag{2.2}$$

For $p=1$, the class A_1 is defined by the condition $Mw(x)\le Cw(x)$ with $[w]_{A_1}=\sup\limits_{x\in\mathbb{R}^n}\frac{Mw(x)}{w(x)}$, and for $p=\infty$, $A_\infty(\mathbb{R}^n)=\cup_{1\le p<\infty}A_p(\mathbb{R}^n)$ and $[w]_{A_\infty}=\inf\limits_{1\le p<\infty}[\omega]_{A_p}$.

Remark 2.1. It is known that

$$w^{1-p'}\in A_{p'/q'}\Rightarrow\left[w^{1-p'}\right]_{A_{p'/q'}(B)}^{p'/q'}=|B|^{-1}\left\|w^{1-p'}\right\|_{L_1(B)}^{q'/p'}\left\|w^{q'/p'}\right\|_{L_{(p'/q')'}(B)}.$$

Moreover, we can write $w^{1-p'}\in A_{p'/q'}\Rightarrow w^{1-p'}\in A_{p'}$ because of $w^{1-p'}\in A_{p'/q'}\subset A_{p'}$. Therefore, we get

$$w^{1-p'}\in A_{\frac{p'}{q'}}\Rightarrow w^{1-p'}\in A_{p'}$$
$$\Rightarrow\left[w^{1-p'}\right]_{A_{p'}(B)}^{\frac{1}{p'}}=|B|^{-1}\left\|w^{1-p'}\right\|_{L_1(B)}^{\frac{1}{p'}}\left\|w^{\frac{1}{p}}\right\|_{L_p(B)}. \qquad (2.3)$$

But the opposite is not true.

Remark 2.2. Let's write $w^{1-p'}\in A_{p'/q'}$ and used the definitions A_p classes we get the following

$$w^{1-p'}\in A_{\frac{p'}{q'}}\Rightarrow\left[w^{1-p'}\right]_{A_{\frac{p'}{q'}}}^{\frac{q(p-1)}{p(q-1)}}=|B|^{-1}\left\|w^{1-p'}\right\|_{L_1(B)}^{\frac{q(p-1)}{p(q-1)}}\left\|w^{\frac{q'}{p}}\right\|_{L_{\left(\frac{p'}{q'}\right)'}(B)}$$
$$\Rightarrow\left[w^{1-p'}\right]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}=|B|^{-\frac{q-1}{q}}\left\|w^{1-p'}\right\|_{L_1(B)}^{\frac{1}{p'}}\|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}}, \qquad (2.4)$$

where the following equalities are provided.

$$1-p'=-\frac{p'}{p},\qquad \frac{q'}{p}=\frac{q}{p(q-1)},\qquad \frac{q'}{p'}=\frac{q(p-1)}{p(q-1)},$$
$$\left(\frac{q}{p}\right)'=\frac{q}{q-p},\left(\frac{p'}{q'}\right)'=\frac{p(q-1)}{q-p}.$$

Then from eq.(2.3) and eq.(2.4) we have

$$w^{1-p'}\in A_{\frac{p'}{q'}}\Rightarrow\left[w^{1-p'}\right]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}$$
$$=|B|^{\frac{1}{q}}\left[w^{1-p'}\right]_{A_{p'}(B)}^{\frac{1}{p'}}\left\|w^{\frac{1}{p}}\right\|_{L_p(B)}^{-1}\|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}} \qquad (2.5)$$

Definition 2.1. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and $1\le p<\infty$. We denote by $M_{p,\varphi}\equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f\in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}}\equiv\|f\|_{M_{p,\varphi}(\mathbb{R}^n)}=sup_{x\in\mathbb{R}^n,r>0}\,\varphi(x,r)^{-1}|B(x,r)|^{-\frac{1}{p}}\|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi}\equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f\in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} \equiv \|f\|_{WM_{p,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}|B(x,r)|^{-\frac{1}{p}}\|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space consisting of all measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} < \infty.$$

Also the spaces $L_p^{loc}(\mathbb{R}^n)$ and $WL_p^{loc}(\mathbb{R}^n)$ endowed with the natural topology are defined as the set of all functions f such that $f\chi_B \in L_p(\mathbb{R}^n)$ and $f\chi_B \in WL_p(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$, respectively.

We define the generalized weighed Morrey spaces as follows.

Definition 2.2. Let $1 \le p < \infty$, $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L_{p,w}^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w\big(B(x,r)\big)^{-\frac{1}{p}}\|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_{p,w}^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w\big(B(x,r)\big)^{-\frac{1}{p}}\|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} =$$

$$\sup_{t>0} t\left(\int_{\{y\in B(x,r):|f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

We say that $\Omega \in Lip_\alpha(S^{n-1})$, $0 < \alpha \le 1$, if there exists a constant $C > 0$ such that $|\Omega(x') - \Omega(y')| \le C|x' - y'|^\alpha$ for all $x', y' \in S^{n-1}$.

The operator μ_Ω was first defined by Stein [22]. Stein proved that if is continuous and satisfies a $Lip_\alpha(S^{n-1})$ $(0 < \alpha \le 1)$ condition, then μ_Ω is an operator of type (p,p) $(0 < p \le 2)$ and of weak type $(1,1)$. In [1], Benedek, Caldero´n and Panzone proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The L_p boundedness of μ_Ω has been studied extensively. See [1, 14, 22, 23], among others. Ding, Fan and Pan [4] proved the weighted

$L_p(\mathbb{R}^n)$ boundedness with A_p weighs for a class of rough Marcinkiewicz integrals. Recently, Ding, Fan and Pan [5] improved the results mentioned above and showed that if $\Omega \in H^1(S^{n-1})$, the Hardy space on the unit sphere, then μ_Ω is still a bounded operator on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. In [24], Xu, Chen and Ying proved the same result as [5] using a different method.

Theorem 2.1. [4, 15] Suppose that Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_s(S^{n-1})$ for some $s \in (1, \infty]$. Then for every $s' < p < \infty$ and $w \in A_{\frac{p}{s'}}(\mathbb{R}^n)$ or $1 < p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}(R^n)$, there is a constant C independent of f such that

$$\|\mu_\Omega(f)\|_{L_{p,w}} \le C\|\Omega\|_{L_s(S^{n-1})}\|f\|_{L_{p,w}}.$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w^* g(t) := \int_t^\infty g(s)w(s)ds, \ 0 < t < \infty,$$

where w is a weight.

The following theorem in the case $w = 1$ was proved in [2].

Theorem 2.2. Let v_1, v_2 and w be weights on $(0, \infty)$ and v_1(t) be bounded outside a neighborhood of the origin. The inequality

$$\operatorname*{ess\,sup}_{t>0} v_2(t)\, H_w^* g(r) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{2.6}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0, \infty)$ if and only if

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{0<\tau<\infty} v_1(\tau)} < \infty. \tag{2.7}$$

Moreover, the value $C = B$ is the best constant for (2.6).

Remark 2.3. In (2.6) and (2.7) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.

3. Marcinkiewicz operator with rough kernels μ_Ω in the spaces $M_{p,\varphi}(w)$

The following Guliyev type local estimates are valid, see [8,9,10].

Lemma 3.1. Suppose that Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$.

If $s' < p < \infty$ and $w \in A_{\frac{p}{s'}}$, then the inequality

$$\|\mu_\Omega f\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0, r)$, and for all $f \in L^{loc}_{p,w}(\mathbb{R}^n)$.

If $1 < p < s$ and $w^{1-p'} \in A_{\frac{p'}{s'}}$, then the inequality

$$\left\| \mu_\Omega f \right\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|^{1/p}_{L_{\frac{s}{s-p}}(B(x_0,r))} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-1/p}_{L_{\frac{s}{s-p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0, r)$, and for all $f \in L^{loc}_{p,w}(\mathbb{R}^n)$.

Proof. Let Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$.

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0, 2r)$. We represent f as

$$f = f_1 + f_2,\ f_1(y) = f(y)\chi_{2B}(y),\ f_2(y) = f(y)\chi_{C_{(2B)}}(y) \tag{3.1}$$

and have

$$\left\| \mu_\Omega(f) \right\|_{L_{p,w}(B)} \leq \left\| \mu_\Omega(f_1) \right\|_{L_{p,w}(B)} + \left\| \mu_\Omega(f_2) \right\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $\mu_\Omega(f_1) \in L_{p,\omega}(\mathbb{R}^n)$ and from the boundedness of μ_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_p(\mathbb{R}^n)$ (see Theorem 2.1) it follows that

$$\left\| \mu_\Omega f_1 \right\|_{L_{p,w}(B)} \leq \left\| \mu_\Omega f_1 \right\|_{L_{p,w}(\mathbb{R}^n)} \lesssim \|\Omega\|_{L_s(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{s'}}} \|f_1\|_{L_{p,w}(\mathbb{R}^n)}$$

$$\approx \|\Omega\|_{L_s(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|f\|_{L_{p,w}(2B)}.$$

It's clear that $x \in B, y \in {}^C(2B)$ implies $\frac{1}{2}|x_0 - y| \leq |x - y| \leq \frac{3}{2}|x_0 - y|$. Then by the Minkowski inequality and conditions on Ω, we get

$$\mu_\Omega(f_2(x)) = \left(\int_0^\infty \left| \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_2(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_0^\infty \left| \int_{B(x,t)} \frac{\Omega(x-y)}{|x_0-y|^{n-1}} f_2(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq C \int_{R^n} \frac{|\Omega(x-y)|}{|x_0-y|^{n-1}} |f_2(y)| dy \left(\int_{|x_0-y|}^\infty \frac{dt}{t^3} \right)^{\frac{1}{2}}$$

$$\leq C \int_{{}^C(2B)} \frac{|\Omega(x-y)| |f(y)|}{|x_0-y|^n} dy.$$

By Fubini's theorem we have

$$\int_{{}^C(2B)} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}dy \approx \int_{{}^C(2B)} |\Omega(x-y)||f(y)| \int_{|x-y|}^{\infty}\frac{dt}{t^{n+1}}dy$$

$$= \int_{2r}^{\infty}\int_{2r\le|x_0-y|<t} |\Omega(x-y)||f(y)|dy\frac{dt}{t^{n+1}} \le C\int_{2r}^{\infty}\int_{B(x_0,t)}|\Omega(x-y)||f(y)|dy\frac{dt}{t^{n+1}}.$$

Therefore,

$$\mu_\Omega(f_2(x)) \le C\int_{2r}^{\infty}\int_{B(x_0,t)}|\Omega(x-y)||f(y)|dy\frac{dt}{t^{n+1}} \ .$$

By applying Holder's inequality for $s' < p < \infty$ and $w \in A_{\frac{p}{s'}}$, we get

$$\int_{{}^C(2B)} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}dy \le C\int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_s(B(x_0,t))}\|f\|_{L_{s'}(B(x_0,t))}\frac{dt}{t^{n+1}}$$

$$\le C\|\Omega\|_{L_s(S^{n-1})}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\left\|w^{-s'/p}\right\|^{1/s'}_{L_{(p/s')'}(B(x_0,t))}\left|B(0,t+|x-x_0|)\right|^{1/s}\frac{dt}{t^{n+1}}$$

$$\le C\|\Omega\|_{L_s(S^{n-1})}[w]^{1/p}_{A_{p/s'}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x,t))}w(B(x_0,t))^{-1/p}|B(x_0,t)|^{1/s'}|B(0,t)|^{1/s}\frac{dt}{t^{n+1}}$$

$$\approx \|\Omega\|_{L_s(S^{n-1})}[w]^{1/p}_{A_{p/s'}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-1/p}\frac{dt}{t}.$$

Moreover, for all $p \in (s', \infty)$ the inequality

$$\|\mu_\Omega(f_2)\|_{L_{p,w}(B)} \le C\|\Omega\|_{L_s(S^{n-1})}[w]^{1/p}_{A_{p/s'}}\, w(B)^{1/p}\int_{2r}^{\infty}\|f\|_{L_{p,ww}(B(x_0,t))}w(B(x_0,t))^{-1/p}\frac{dt}{t}$$

is valid. Thus

$$\|\mu_\Omega(f)\|_{L_{p,w}(B)} \le C\|\Omega\|_{L_s(S^{n-1})}[w]^{1/p}_{A_{p/s'}}$$

$$\times\left(\|f\|_{L_{p,w}(2B)} + w(B)^{1/p}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-1/p}\frac{dt}{t}\right).$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B|\|f\|_{L_{p,w}(2B)}\int_{2r}^{\infty}\frac{dt}{t^{n+1}}$$

$$\le C|B|\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\frac{dt}{t^{n+1}}$$

$$\le Cw(B)^{\frac{1}{p}}\left\|w^{-1/p}\right\|_{L_{p'}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\frac{dt}{t^{n+1}}$$

$$\leq C w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \left\|w^{-1/p}\right\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{n+1}}$$

$$\leq C [w]_{A_{\frac{p}{s'}}}^{\frac{1}{p}} w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}.$$

Thus

$$\|\mu_\Omega(f)\|_{L_{p,w}(B)} \leq C \|\Omega\|_{L_s(S^{n-1})} [w]_{A_{p/s'}}^{1/p} w(B)^{1/p} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-1/p} \frac{dt}{t}.$$

Let also $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$. Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $\mu_\Omega(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of μ_Ω in $L_{p,w}(R^n)$ for $w^{1-p'} \in A_{p'/s'}$ and $1 < p < s$ (see Theorem 2.1) it follows that

$$\|\mu_\Omega(f_1)\|_{L_{p,w}(B)} \leq \|\mu_\Omega(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \leq C \|\Omega\|_{L_s(S^{n-1})} [w^{1-p'}]_{A_{\frac{p'}{s'}}}^{\frac{1}{p'}} \|f_1\|_{L_{p,w}(\mathbb{R}^n)}$$

$$\approx \|\Omega\|_{L_s(S^{n-1})} \left[w^{1-p'}\right]_{A_{\frac{p'}{s'}}}^{\frac{1}{p'}} \|f\|_{L_{p,w}(2B)}.$$

If $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$, then using (3.2), Minkowski theorem and Hölder inequality,

$$\|\mu_\Omega(f_2)\|_{L_{p,w}(B)} \leq \left(\int_B \left(\int_{2r}^{\infty} \int_{B(x_0,t)} |\Omega(x-y)| |f(y)| dy \frac{dt}{t^{n+1}} \right)^p w(x) dx \right)^{\frac{1}{p}}$$

$$\leq \int_{2r}^{\infty} \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_{p,w}(B)} |f(y)| dy \frac{dt}{t^{n+1}}$$

$$\leq C \int_{2r}^{\infty} \int_{B(x_0,t)} \|\Omega(\cdot - y)\|_{L_s(B)} \|w\|_{L_{(s/p)'}(B)}^{\frac{1}{p}} |f(y)| dy \frac{dt}{t^{n+1}}$$

$$\leq C \|\Omega\|_{L_s(S^{n-1})} \|w\|_{L_{(s/p')'}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \int_{B(x_0,t)} |B(0, r + |x_0 - y|)|^{\frac{1}{s}} |f(y)| dy \frac{dt}{t^{n+1}}$$

$$\leq C \|\Omega\|_{L_s(S^{n-1})} \|w\|_{L_{(s/p')'}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_1(B(x_0,t))} |B(0, r+t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}}$$

$$\leq C \|\Omega\|_{L_s(S^{n-1})} \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \left\|w^{-p'/p}\right\|_{L_1(B(x_0,t))}^{1/p'} |B(x_0,t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}}$$

$$\leq C \|\Omega\|_{L_s(S^{n-1})} |B|^{\frac{1}{s}} \|w\|_{L_{\frac{s}{s-p}}(B)}^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \left\|w^{1-p'}\right\|_{L_1(B(x_0,t))}^{1/p'} |B(x_0,t)|^{\frac{1}{s}} \frac{dt}{t^{n+1}}$$

is obtained. By applying (2.3) for $\|w^{1-p'}\|^{1/p'}_{L_1(B(x_0,t))}$ and (2.5) for $\|w\|^{1/p}_{L_{\frac{s}{s-p}}(B)}$ we have the following inequality

$$\|\mu_\Omega(f_2)\|_{L_{p,w}(B)} \le C\|\Omega\|_{L_s(S^{n-1})}\left|w^{1-p'}\right|^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}}\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\|w\|^{-\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\frac{dt}{t}$$

is valid. Thus

$$\|\mu_\Omega(f)\|_{L_{p,w}(B)} \le C\|\Omega\|_{L_s(S^{n-1})}\left[w^{1-p'}\right]^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}} \times\left(\|f\|_{L_{p,w}(2B)} + \|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\|w\|^{-\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\frac{dt}{t}\right).$$

On the other hand,

$$\begin{aligned}\|f\|_{p,w(2B)} &\approx |B|\,\|f\|_{L_{p,w}(2B)}\int_{2r}^{\infty}\frac{dt}{t^{n+1}}\\ &\le C|B|\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\frac{dt}{t^{n+1}}\\ &= \left[w^{1-p'}\right]^{-\frac{1}{p'}}_{A_{p'}(B)}|B|^{\frac{1}{s}}\left\|w^{1-p'}\right\|^{\frac{1}{p'}}_{L_1(B)}\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\frac{dt}{t^{n+1}}\\ &\le C\left[w^{1-p'}\right]^{-\frac{1}{p'}}_{A_{p'}(B)}\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}|B(x_0,t)|^{\frac{1}{s}}\left\|w^{1-p'}\right\|^{\frac{1}{p'}}_{L_1(B(x_0,t))}\frac{dt}{t^{n+1}}\\ &\le C\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\|w\|^{-\frac{1}{p'}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\frac{dt}{t}.\end{aligned}$$

Thus

$$\|\mu_\Omega(f)\|_{L_{p,w}(B)} \le C\|\Omega\|_{L_s(S^{n-1})}\left[w^{1-p'}\right]^{\frac{1}{p'}}_{A_{\frac{p'}{s'}}}\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\|w\|^{-\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x_0,t))}\frac{dt}{t}.$$

Thus we complete the proof of Lemma 3.1.

Theorem 3.1. Suppose that Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$. Let $s' < p < \infty$, $w \in A_{p/s'}(\mathbb{R}^n)$ and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^{\infty}\frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}}\frac{dt}{t} \le C\varphi_2(x,r) \tag{3.4}$$

where C does not depend on x and r. Let also, $1 < p < s$, $w^{1-p'} \in A_{p'/q'}$ and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\|w\|_{L_{\frac{s}{s-p}}(B(x,\tau))}^{\frac{1}{p}}}{\|w\|_{L_{\frac{s}{s-p}}(B(x,t))}^{\frac{1}{p}}} \frac{dt}{t} \le C\varphi_2(x,r) \frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|_{L_{\frac{s}{s-p}}(B(x,r))}^{\frac{1}{p}}}, \tag{3.5}$$

where C does not depend on x and r.

Then the operator μ_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $p>1$. Moreover

$$\|\mu_\Omega(f)\|_{M_{p,\varphi_2}(w)} \le C\|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. By Lemma 3.1 and Theorem 2.2 with $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $\nu_2(r) = \varphi_2(x,r)^{-1}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}}$ we have

$$\begin{aligned}\|\mu_\Omega(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_\Omega f\|_{L_{p,w}(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\ &= \|f\|_{M_{p,\varphi_1}(w)}.\end{aligned}$$

Remark 3.1. Note that Lemma 3.1 and Theorem 3.3 in the case $s=\infty$ was proved in [12].

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References

1. Benedek A., Calderon A.P., Panzone R., Convolution operators on Banach value functions, Proc. Natl. Acad. Sci. USA, Vol.48, (1962), pp.256-26.
2. Burenkov V.I., Gogatishvili A., Guliyev V.S., Mustafayev R.Ch., Boundedness of the Riesz potential in local Morrey-type spaces, Potential Anal. Vol.35, No.1, (2011), pp.67-87.
3. Deringoz F., Parametric Marcinkiewicz integral operator and its higher order commutators on generalized weighted Morrey spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Vol.37, No.4, (2017), pp.24-32.
4. Ding Y., Fan D., Pan Y., Weighted boundedness for a class of rough Marcinkiewicz integrals, Indiana Univ. Math. J., Vol.48, (1999), pp. 1037-1055.

5. Ding Y., Fan D., Pan Y., L_p-boundedness of Marcinkiewicz integrals with Hardy space function kernels, Acta Math. Sinica, English Series, Vol.16, (2000), pp.593-600.
6. Ding Y., Lu S., Yabuta K., On commutators of Marcinkiewicz integrals with rough kernel, J. Math. Anal. Appl., Vol.275, (2002), pp.60-68.
7. Duoandikoetxea J., Weighted norm inequalities for homogeneous singular integrals, Trans. Amer. Math. Soc., Vol.336, (1993), pp.869-880.
8. Guliyev V.S., Integral operators on function spaces on the homogeneous groups and on domains in Rn. Doctoral dissertation, Moscow, Mat. Inst. Steklov, (1994), 329 p. (in Russian)
9. Guliyev V.S., Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, J. Inequal. Appl., (2009), Art. ID 503948, 20 p.
10. Guliyev V.S., Generalized weighted Morrey spaces and higher order commutators of sublinear operators, Eurasian Math. J., Vol.3, No.3, 2012, pp. 33-61.
11. Guliyev V.S., Karaman T., Mustafayev R.Ch., Serbetci A., Commutators of sublinear operators generated by Calder´on-Zygmund operator on generalized weighted Morrey spaces, Czechoslovak Math. J., Vol. 64, No.2, 2014, pp.365-386.
12. Guliyev V.S., Ismayilova A.F., Marcinkiewicz integral on generalized weighted Morrey spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Vol.34, No.4, (2014), pp.53-64.
13. Hamzayev V.H., Maximal operator with rough kernel and its commutators in generalized weighted Morrey spaces, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Vol.40, No.1, (2020), pp.96-110.
14. Hörmander L., Translation invariant operators, Acta Math. Vol.104, (1960), pp.93-139.
15. Guoen Hu, Meng Qu, Quantitative weighted L_p bounds for the Marcinkiewicz integral, Math. Inequal. Appl., Vol.22, No.3, (2019), pp.885-899.
16. Komori Y., Shirai S., Weighted Morrey spaces and a singular integral operator, Math. Nachr., Vol.282, No.2, 2009, pp.219-231.
17. Mizuhara T., Boundedness of some classical operators on generalized Morrey spaces, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, (1991), pp.183-189.
18. Morrey C.B., On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., Vol.43, (1938), pp.126-166.
19. Muckenhoupt B., Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc., Vol.165, (1972), pp.207-226.
20. Nakai E., Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces, Math. Nachr., Vol.166, (1994), pp.95-103.
21. Sawano Y. A thought on generalized Morrey spaces, J. Indonesian Math. Soc., Vol.25, No.3, (2019), pp.210-281.

22. Stein E.M., On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz, Trans. Amer. Math. Soc., Vol.88, (1958), pp.430-466.
23. Walsh T., On the function of Marcinkiewicz, Studia Math., Vol.44, 1972, pp.203-217.
24. Xu H., Chen J.C., Ying Y.M., A note on Marcinkiewicz integrals with H1 kernels, Acta. Math. Scientia, Ser. B, Vol. 23, No. 1, 2003, pp.133-138.